



Munich Personal RePEc Archive

# Analysis of Reciprocity and Substitution Theorems, and Slutsky Equation

Haradhan Mohajan

Assistant Professor, Premier University, Chittagong, Bangladesh.

14 January 2017

Online at <https://mpra.ub.uni-muenchen.de/82938/>

MPRA Paper No. 82938, posted 27 November 2017 02:17 UTC

# Analysis of Reciprocity and Substitution Theorems, and Slutsky Equation

Haradhan Kumar Mohajan\*

\*Premier University, Chittagong, Bangladesh

**Abstract:** The mathematical techniques are considered here to explain the economical problems. This article deals with the Substitution and Reciprocity Theorems for the various commodities. Finally, it also has considered the Slutsky Equation for the minimization of the prices and the budget constraints. The study has included analysis of some explicit examples to clarify the concepts of the results. The aim of this paper is to make the mathematical concepts interesting to the economists. An attempt has been taken here to discuss the problems in some detailed mathematical analysis in an elegant manner.

**Keywords:** Indifference Hypersurface, Lagrange Multiplier, Utility Function, Reciprocity and Substitution Theorems, Slutsky Equation.

## 1. Introduction

In this study we have considered the Substitution and Reciprocity Theorems concerned with commodities, and related matters such as, commodities which are substitutes and those which are complements of each other. Here we also consider one of the related results, the Slutsky Equation, which is concerned with the vector that minimizes utility in terms of changes the price and the budget constraint. To give the clear idea of some of these results, we have dealt with some explicit examples.

The Slutsky Equation ([Slutsky, 1915](#)) has a long and recognized history in microeconomics. It was first expressed in mathematical economics in 1915 by Russian statistician and economist Eugene Slutsky (1880–1948) ([Chipman and Lenfant, 2002](#)). The equation relates changes in Marshallian uncompensated price to the changes in Hicksian compensated price. J. R. Hicks conjectured that Slutsky symmetry should hold for discrete as well as infinitesimal price changes if demand functions are globally linear ([Hicks, 1956](#)). The equation has laid the foundation for rigorous analysis of optimal consumption decision in microeconomics ([Nicholson, 2005](#)). At present the Slutsky Equation is a staple of most modern microeconomics and also a pioneer topic of future research ([Varian, 2003](#); [Weber, 2002](#)).

Perhaps P. A. Samuelson is one of the first who has argued that Slutsky's symmetry result seems to apply only for differential size changes in prices ([Samuelson, 1947](#)). R. C. Geary has shown that if demand curves are to be linear in income and the prices of other goods, then the utility function must take the specific functional form ([Geary, 1950](#)). C. Weber proves Hicks conjecture using the linear expenditure system utility function and the Slutsky compensation for price changes ([Weber, 2002](#)).

In section 2 we give mathematical notations, in section 3 we provide indifference curves and hypersurfaces, and in section 4 we briefly describe price vector and budget constraint. Further, in section 5 we have included Substitution Theorem and section 6 contains Reciprocity Theorem. Section 7 provides Substitution and Reciprocity Theorems with Lagrange multipliers. Section 8 presents the Slutsky Equation, and conclusion is offered in section 9.

## 2. Mathematical Notations

We consider an individual in an  $n$ -commodity/good world. Suppose two bundles of commodities are

$$\mathbf{x} = (x_1, x_2, \dots, x_n) \quad \mathbf{y} = (y_1, y_2, \dots, y_n)$$

represented by the vectors  $\mathbf{x}$  and  $\mathbf{y}$  in  $n$ -dimensional Euclidean space

$R^n$   $\mathbf{x} \geq \mathbf{y} \Rightarrow x_i \geq y_i$   $\mathbf{x} > \mathbf{y} \Rightarrow \mathbf{x} \geq \mathbf{y}$  but  $\mathbf{x} \neq \mathbf{y}$   $x_i$   $y_i$   
, then  $\mathbf{x} \gg \mathbf{y} \Rightarrow x_i > y_i$  for all  $i$ ; , that is,  $\mathbf{x}$  is different from  $\mathbf{y}$  for at least

one  $i$ ; and  $\mathbf{x} \gg \mathbf{y} \Rightarrow x_i > y_i$  for all  $i$  ([Islam, 1997](#)).

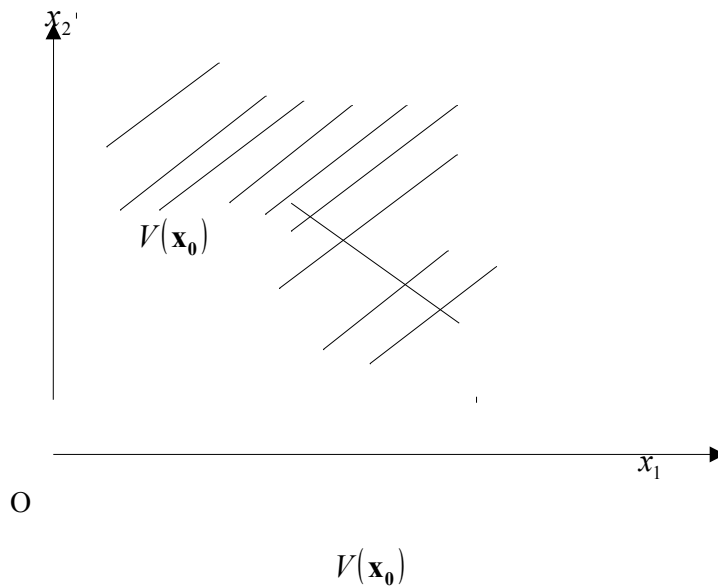
The components represent amounts of different commodities in some unit, such as kilogram. We assume that one prefers the bundle  $\mathbf{x}$  to the bundle  $\mathbf{y}$ , then we can write it as;  $\mathbf{x} P \mathbf{y}$ . We use the notation  $\mathbf{x} R \mathbf{y}$  to mean that either  $\mathbf{x}$  is preferred to  $\mathbf{y}$  or  $\mathbf{x}$  is indifferent to  $\mathbf{y}$ . We now define the  $n$ -dimensional utility function (a single hypersurface) as ([Islam et al., 2009a; 2009b](#)).

$$u(\mathbf{x}) = u(x_1, x_2, \dots, x_n) = c \quad (1)$$

for some fixed constant  $c$ .

In preference relation we can write;

$$u(\mathbf{x}) > u(\mathbf{y}) \Leftrightarrow \mathbf{x}P\mathbf{y} \quad (2)$$



**Figure 1:** The convex set  $V(\mathbf{x}_0)$  is the shaded region.

Let us consider a fixed vector  $\mathbf{x}_0$ , and consider the set of all the vectors  $\mathbf{x}$  which are preferred to  $\mathbf{x}_0$ .  
If we denote this set by  $V(\mathbf{x}_0)$ , we can write ([Cassels, 1981](#));

$$V(\mathbf{x}_0) = \{\mathbf{x} : \mathbf{x}P\mathbf{x}_0\} \quad (3)$$

For the utility function it can be written as,

$$V(\mathbf{x}_0) = \{\mathbf{x} : u(\mathbf{x}) > u(\mathbf{x}_0)\} \quad (3a)$$

where  $V(\mathbf{x}_0)$  is a convex set (Figure 1).

### 3. Indifference Curves and Hypersurfaces

Indifference curves (ICs) were first introduced by the English economist F. M. Edgeworth in 1880s. The concept was redefined and used extensively by the Italian economist Vilfredo Pareto in the early 1900s. ICs are popularized and greatly extended in the 1930s by two other English economists R. G. Allen and John R. Hicks. ICs are crucial tool of analysis because they are used to represent an ordinal measure of the tastes and preferences of the consumers and to show how the consumer maximizes utility in spending income ([Mohajan, 2017](#)).

Let us consider the simple form for utility function, for  $n = 2$ ;

$$u(\mathbf{x}) = x_1 x_2 \quad (4)$$

Consider now the curves in the  $x_1 x_2$ -plane are given by;

$$u(x_1, x_2) = c \quad u(x_1, x_2) = c_1 \quad u(x_1, x_2) = c_2 \quad (5)$$

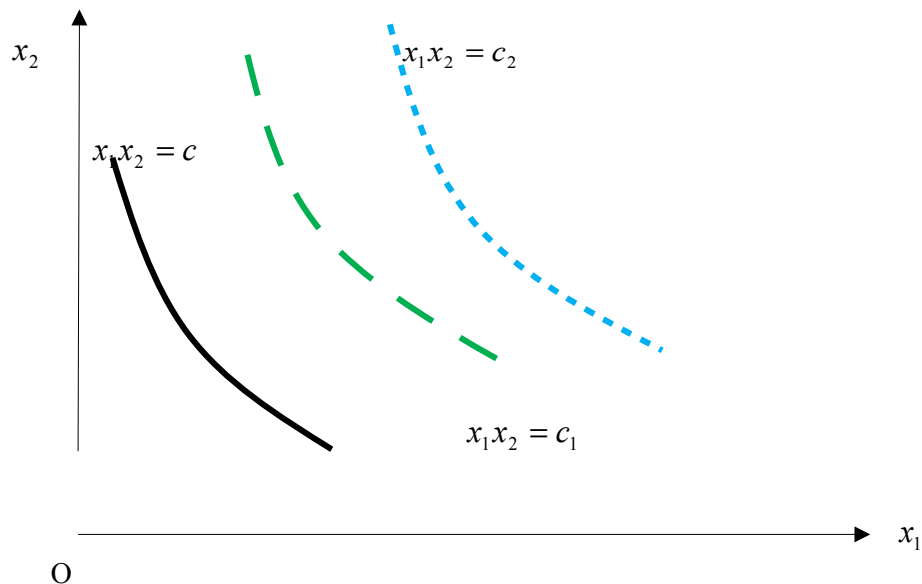
where  $0 < c < c_1 < c_2$  (say). Combining (4) and (5) we get;

$$x_1 x_2 = c \quad x_1 x_2 = c_1 \quad x_1 x_2 = c_2 \quad (6)$$

are rectangular hyperbolae (Figure 2). In three-dimensional case (6) will be as follows:

$$x_1 x_2 x_3 = c \quad x_1 x_2 x_3 = c_1 \quad x_1 x_2 x_3 = c_2 \quad (7)$$

and are called rectangular hyperboloid. Consider all 'points' or 'bundles' lie on the curve  $x_1 x_2 = c$  (Figure 2). Since,  $u(\mathbf{x}) = x_1 x_2$ , all these points have the same value for  $u(x_1, x_2)$ , namely, the value 'c'. Similarly, all the points on the curve  $x_1 x_2 = c_1$  have the same value for  $u(x_1, x_2)$ , namely, the value  $c_1$ .



**Figure 2:** The rectangular hyperbolae (6) lying in the positive quadrant with  $0 < c < c_1 < c_2$ .

But,  $c_1 > c$ , and so if  $(x_1, x_2)$  is any point on the curve  $x_1 x_2 = c$ , and  $(y_1, y_2)$  is any point on the curve  $x_1 x_2 = c_1$ , that is, if  $y_1 y_2 = c_1$ , then we have  $u(y) = c_1 > u(x) = c$ , and so that  $y P x$ . Thus, all the points on the curve  $x_1 x_2 = c_1$  are preferred to all the points on the curve  $x_1 x_2 = c$ , and all the points on the curve  $x_1 x_2 = c_2$  are preferred to all the points on the curve  $x_1 x_2 = c_1$  and so on. Hence, all the points on the curve  $x_1 x_2 = c$  are clearly equivalent to each other, since they give the same value of the utility function, namely  $c$ . In other words, the individual is indifferent to the bundles represented by points on the same curve. These types of curves are called indifference curves. ICs do not intersect each other's. Then the set (7) is more appropriately generalized as;

$$\hat{V}(x') = \{x : x R x'\} \quad (8)$$

that is, the set  $x$ , which are preferred to or indifferent to  $x'$ . So that;

$$x R y \Leftrightarrow u(x) \geq u(y) \quad (9)$$

Let,  $x' = (x'_1, x'_2)$  be a fixed point or bundle. The indifference curve on which this point lies is given by;

$$x_1 x_2 = \text{constant} = x'_1 x'_2 \quad (10)$$

since  $(x'_1, x'_2)$  must satisfy the equation of this curve. From (8) and (10) it is clear that the set consists of all points  $x$  such that (Vigier, 2012).

$$u(x) \geq u(x') \quad x_1 x_2 \geq x'_1 x'_2 \quad (11)$$

, that is,

This set consists of points lying on the right and above the curve (11), that is, in the shaded region  $\hat{V}(x')$  as like as in Figure 2. So that,  $\hat{V}(x')$  is of course convex (Islam et al., 2009a).

By a *hypersurface* we mean the set of points in  $n$ -dimensional Euclidean space  $R^n$  for which  $f(x) =$

constant. For different values of the constant, we find corresponding different hypersurfaces. For  $n = 3$  we have different surfaces, on the other hand for  $n = 2$  we have simply curves. The indifference hypersurfaces do not intersect each other in the finite region. Since all the components of the vectors are non-negative so, we will deal here only with non-negative coordinates. For  $n = 2$ , the curves lie in the first quadrant and for  $n = 3$ , the hypersurfaces lie in the first octant (Mohajan, 2017).

## 4. Price Vector and Budget Constraint

We consider a bundle of two commodities, so that  $(x_1, x_2)$  represents a bundle of  $x_1$  kg of rice, and  $x_2$  kg of wheat (say). Let,  $p_1$  be the cost of 1 kg of rice, and  $p_2$  be the cost of 1 kg of wheat in dollar. We call  $\mathbf{p} = (p_1, p_2)$  the price vector of possible bundles of rice and wheat. The total cost of the bundle of commodities  $x_1$  and  $x_2$  is ([Mohajan, 2017](#)),

$$p_1x_1 + p_2x_2 = \mathbf{p} \cdot \mathbf{x} \quad (12)$$

where  $\mathbf{p} \cdot \mathbf{x}$  is a scalar product of vectors  $\mathbf{p}$  and  $\mathbf{x}$ .

We now introduce the idea of a budget constraint. For bundle  $\mathbf{x}$  with a price vector  $\mathbf{p}$  let us consider one has maximum  $c$  amount of dollars to spend, then we can write,

$$\mathbf{p} \cdot \mathbf{x} \leq c \quad (13)$$

which is referred to as budget constraint. Let us consider the hypersurfaces,

$$u(\mathbf{x}) = \text{Constant}, \quad (14)$$

for various values of the constant. According to (6) the individual concerned is indifferent to the bundles represented by all these vectors, that is, all these bundles for him/her are 'equally good' (or 'equally bad'). That is why (6) and (7) are indifferent hypersurfaces. For simplicity we consider  $n = 2$ , so ([Leung and Sproule, 2007](#)),

$$u(\mathbf{x}) = x_1x_2 \quad (15)$$

The indifference curves are given by rectangular hyperbolae,

$$x_1x_2 = k \quad (16)$$

where,  $k = \text{constant} > 0$ . Let the fixed price vector be  $\mathbf{p} = (p_1, p_2)$  then by (13) the budget constraint is,

$$p_1x_1 + p_2x_2 \leq c \quad (17)$$

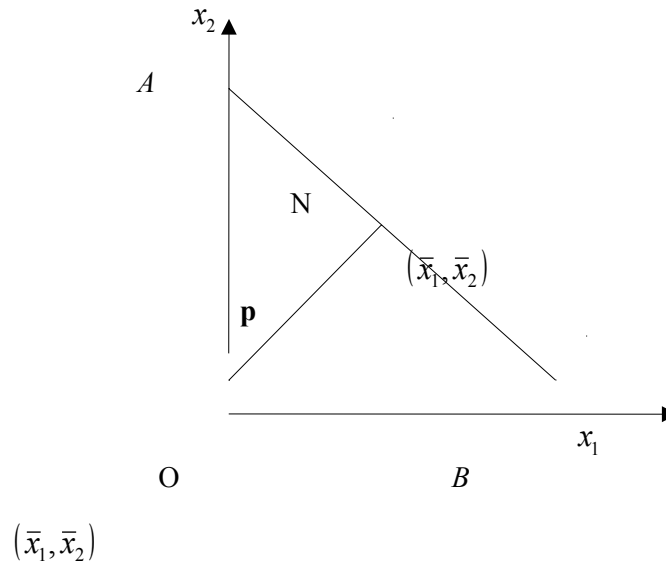
with fixed  $c$ . If we draw a straight line  $(AB)$ ,

$$p_1x_1 + p_2x_2 = c \quad (18)$$

then, there is only one member of family of ICs (16) that touches the straight line (18). Let, it touches at the point  $(\bar{x}_1, \bar{x}_2)$  which is a vector and it maximizes the utility (Figure 3). The inequality (17)

restricts  $(\bar{x}_1, \bar{x}_2)$  to the interior or boundary of the  $\triangle OAB$ , where,  $ON$  is parallel to the vector  $\mathbf{p}$ . The maximum of the utility function must occur on the line  $AB$  but, not in the interior of  $\triangle OAB$ .

$$ON = \left| \frac{c}{\sqrt{p_1^2 + p_2^2}} \right|$$



**Figure 3:** The point  $(\bar{x}_1, \bar{x}_2)$  maximizes the utility.  $ON$  is parallel to price vector  $\mathbf{p}$  which is perpendicular to  $AB$ .

The indifference curve which gives the maximum is ([Islam et al., 2009b](#)),

$$x_1 x_2 = \bar{x}_1 \bar{x}_2 \equiv \frac{c^2}{4p_1 p_2} \quad (19)$$

$$x_2 = \frac{c^2}{4p_1 p_2 x_1} \quad p_1 x_1 + \frac{c^2}{4p_1 x_1} = c$$

From (19), we get; and substituting in (18) we find; whose discriminant  $x_1 = x_2$   $(\bar{x}_1, \bar{x}_2)$  is zero, so (18) has two common roots, and hence, the curve and the line touch at a point.

## 5. Substitution Theorem

Substitution Theorem states that an economy with many commodities but only one factor input, say labor, will not substitute inputs, for example, commodities or labor, when final demand is changed. It claims that the entire production possibility frontier is achievable through one technology ([Raa, 1995](#)).

$$\mathbf{z}(\mathbf{p}) = (z_1(\mathbf{p}), z_2(\mathbf{p}), \dots, z_n(\mathbf{p}))$$

Let us consider the vector which minimizes the cost  $\mathbf{p} \cdot \mathbf{x}$  as; so that  $\mathbf{p} \cdot \mathbf{x} \geq (\mathbf{p} \cdot \mathbf{z}(\mathbf{p}))$  for all  $\mathbf{x}$  lying on the hypersurface (1). Again we consider two distinct price vectors  $\mathbf{p}'$

$\mathbf{p}''$  and  $\mathbf{p}'' \cdot \mathbf{x}$ , and write  $\mathbf{z}' = \mathbf{z}(\mathbf{p}')$ ,  $\mathbf{z}'' = \mathbf{z}(\mathbf{p}'')$ , that is, the vectors  $\mathbf{z}'$  and  $\mathbf{z}''$  minimize the total cost  $\mathbf{p}' \cdot \mathbf{x}$  and  $\mathbf{p}'' \cdot \mathbf{x}$  respectively, on the indifference hypersurface (1). Now, we can write ([Islam, 1997](#)).

$$\mathbf{p}' \cdot \mathbf{x} \geq \mathbf{p}' \cdot \mathbf{z}' \quad \mathbf{p}'' \cdot \mathbf{x} \geq \mathbf{p}'' \cdot \mathbf{z}'' \quad ; \quad (20)$$

for all vectors  $\mathbf{x}$  lying on (1). Since the vectors  $\mathbf{z}'$  and  $\mathbf{z}''$  lie on the hypersurface (1), we are free to put  $\mathbf{x} = \mathbf{z}''$  in the first inequality, and to put  $\mathbf{x} = \mathbf{z}'$  in the second inequality in (20), then we get the following inequalities;

$$\mathbf{p}' \cdot \mathbf{z}'' \geq \mathbf{p}' \cdot \mathbf{z}' \quad \mathbf{p}'' \cdot \mathbf{z}' \geq \mathbf{p}'' \cdot \mathbf{z}'' \quad ; \quad (21)$$

$$\Rightarrow 0 \geq \mathbf{p}' \cdot (\mathbf{z}' - \mathbf{z}'') \quad 0 \geq \mathbf{p}'' \cdot (\mathbf{z}'' - \mathbf{z}')$$

$$\Rightarrow 0 \geq (\mathbf{p}' - \mathbf{p}'') \cdot (\mathbf{z}' - \mathbf{z}'') \quad (22)$$

which is known as Substitution Theorem ([Islam et al., 2009b](#)). To obtain a differential form of (22) let the vectors  $\mathbf{p}'$  and  $\mathbf{p}''$  differ only in their  $i^{\text{th}}$  component;

$$p'_i \neq p''_i \quad p'_i = p''_j, \quad i \neq j$$

i.e., and .

Now we can write (22) as;

$$0 \geq (p'_1 - p''_1)(z'_1 - z''_1) + \dots + (p'_i - p''_i)(z'_i - z''_i) + \dots + (p'_n - p''_n)(z'_n - z''_n)$$

$$\text{Here } (p'_1 - p''_1), \dots, (p'_n - p''_n) \text{ all vanishes except } (p'_i - p''_i)$$

$$0 \geq (p'_i - p''_i)(z'_i - z''_i)$$

$$0 \geq (p'_i - p''_i)(z'_i(\mathbf{p}') - z_i(\mathbf{p}'')) \quad (23)$$

Here  $p'_1 = p''_1$ , etc., and subscript  $i$  refers to a single term and hence, should not be summed over. From (23) we observe that each component of  $z$ , for example,  $z_i$  depends on all components of  $\mathbf{p}'$ . From (23) we see that  $z_i(\mathbf{p})$  is a decreasing function of  $p_i$ , i.e., if  $p''_i > p'_i$  then  $z_i(\mathbf{p}'') < z_i(\mathbf{p}')$  and vice-versa.



Now we consider  $p'_i$  and  $p''_i$  differ infinitesimally, i.e.,  $p''_i - p'_i > 0$ , i.e.,  $dp_i > 0$  and  $p''_i = p'_i + dp'_i$  then ((23) becomes,

$$0 \geq (-dp'_i)(z_i(p'_1, \dots, p'_i, \dots)) - (z_i(p'_1, \dots, p'_i + dp'_i, \dots)) \quad (24)$$

and the components of  $z_i$  differ only in the  $i^{\text{th}}$  component  $p'_i$ . Now applying Taylor series in (24) we get;

$$\begin{aligned} & z_i(p'_1, \dots, p'_i + dp'_i, \dots) \\ &= z_i(p'_1, \dots, p'_i, \dots) + \frac{\partial z_i}{\partial p'_i} dp'_i + \dots \end{aligned} \quad (25)$$

Using (25) we can write (24) as;

$$0 \geq (-dp'_i) \left( \frac{\partial z_i}{\partial p'_i} dp'_i \right) \Rightarrow (dp'_i)^2 \left( \frac{\partial z_i}{\partial p'_i} \right) \leq 0 \Rightarrow \frac{\partial z_i}{\partial p'_i} \leq 0 \quad (dp'_i)^2 > 0$$

, since , where no summation is imposed on  $i$ . Now dropping the prime we get;

$$\frac{\partial z_i}{\partial p_i} \leq 0 \quad (26)$$

Let,  $\mathbf{m}$  be any arbitrary vector in (22) and let us consider  $p''_i = p'_i + \delta \mathbf{m}$ , where  $\mathbf{m}$  is infinitesimal quantity. We get from (22);  $0 \geq -\delta(\mathbf{m})(z(\mathbf{p}') - z(\mathbf{p}' + \delta \mathbf{m}))$ . With the use of a suitable modification of

Taylor expansion we get;  $0 \geq \sum_{i,j} \delta^2 \left( m_i \frac{\partial z_i(\mathbf{p})}{\partial p'_j} m_j \right) \delta^2 > 0$ , since and dropping prime we get;

$$\sum_{i,j} m_i m_j \frac{\partial z_i(\mathbf{p})}{\partial p_j} \leq 0, \forall \mathbf{m} \quad (27)$$

This is the Substitution Theorem in differential form.

## 6. Reciprocity Theorem

Reciprocity means that people reward for kind actions and punish for unkind ones. It is a powerful determinant of human behavior (Falk and Fischbacher, 2000). Reciprocity plays an important role in several economic interactions because, the expectation that future benefits will be distributed by another agent (Stanca, 2007). According to Stuart Ballantine, Reciprocity Theorem is, “Among the tools of thought and artifices by which man forces his mind to give him more service, perhaps the most intensely useful are the simple mathematical rules of inversion known as Reciprocity Theorems” (Ballantine, 1928).

Now we will discuss the mathematical formulation of Reciprocity Theorem in some detail with examples. Let, the price vector  $\mathbf{p}$  be changed by an infinitesimal amount  $d\mathbf{p}$  and denotes the corresponding

change in  $\mathbf{z}(\mathbf{p})$  by  $d\mathbf{z}(\mathbf{p})$ . Since  $\mathbf{z}(\mathbf{p})$  and  $\mathbf{z}(\mathbf{p})+d\mathbf{z}(\mathbf{p})$  are situated on the same indifference hypersurface (1);  $\mathbf{z}+d\mathbf{z}$ , in the limit, is in the tangent plane at  $\mathbf{z}$ , at which the vector  $\mathbf{p}$  is normal, so that;

$$\mathbf{p}.d\mathbf{z}=0. \quad (28)$$

Let  $r(\mathbf{p})$  be the cost of the bundle that minimizes the cost on (1), so that;  $r(\mathbf{p}) = \mathbf{p}.\mathbf{z}(\mathbf{p})$ . Taking differentials we get;

$$dr(\mathbf{p}) = \mathbf{p}.d\mathbf{z}(\mathbf{p}) + d\mathbf{p}.\mathbf{z}(\mathbf{p}) = d\mathbf{p}.\mathbf{z}(\mathbf{p}) \quad , \text{ by (28).} \quad (29)$$

The left side of (29) is perfect differential and we can write it as;

$$dr(\mathbf{p}) = \frac{\partial r}{\partial p_i} dp_i = z_i(\mathbf{p}) dp_i \quad (30)$$

which is true for all  $dp_i$ , so that we get;

$$z_i(\mathbf{p}) = \frac{\partial r}{\partial p_i} \quad (31)$$

$$\frac{\partial z_i}{\partial p_j} = \frac{\partial^2 r}{\partial p_j \partial p_i} = \frac{\partial}{\partial p_i} \left( \frac{\partial r}{\partial p_j} \right) = \frac{\partial z_j}{\partial p_i}$$

Hence,  $\frac{\partial z_i}{\partial p_j} = \frac{\partial z_j}{\partial p_i}$ , by (31). (32)

This is the Reciprocity Theorem in differential form. Now we assume that;

$$\frac{\partial z_i}{\partial p_j} = \frac{\partial z_j}{\partial p_i} > 0 \quad (33)$$

which may not necessarily be the case. We will see that it would not be inappropriate, in this case, for

example, to take the commodity  $x_i$  to represent tea, and the commodity  $x_j$  for coffee. Now  $z_i(\mathbf{p})$  and  $z_j(\mathbf{p})$

are respectively the commodities which minimize the total cost at price  $\mathbf{p}$  of all the bundles in the indifference hypersurface (1). We have mentioned that  $z_i(\mathbf{p})$  and  $z_j(\mathbf{p})$ , and also all other components of  $\mathbf{z}(\mathbf{p})$   $p_i$   $p_j$

, each depends on all components of  $\mathbf{p}$ . In this particular case,  $p_i$  and  $p_j$  are prices of unit amounts of tea and coffee respectively. Equation (33) indicates that if rate of increase of the amount of tea that minimizes the total cost as the price goes up, is the same as that of the amount of coffee that minimizes the total cost as the price of tea goes up. We can interpret this situation such that the two commodities in this case are substitutes. Hence, if the price of tea goes up we use more coffee, and vice versa ([Islam et al., 2009b](#)). If the situation may be such that,

$$\frac{\partial z_i}{\partial p_j} = \frac{\partial z_j}{\partial p_i} < 0 \quad , \quad (34)$$

then, similar reason as above we can say that commodities  $x_i$  and  $x_j$  are tea and sugar respectively. Hence, if the price of tea goes up, we use less sugar and vice versa. To derive a result about the relative occurrence of substitutes and complements, consider the relation (16) is valid for all commodities;

$$\frac{\partial z_i}{\partial p_i} \leq 0 \quad , i=1, 2, \dots, n. \quad (35)$$

From (28) we get;

$$\mathbf{p} \cdot d\mathbf{z} = \sum_i p_i dz_i = \sum_{i,k} p_i \frac{\partial z_i}{\partial p_k} dp_k \quad . \quad (36)$$

We have considered the increments  $dp_k$  to be independent, so we can write;

$$\sum_i p_i \frac{\partial z_i}{\partial p_k} = 0 \quad , k=1, 2, \dots, n. \quad (37)$$

For some particular value of  $k$  we can write the relation (37) as;

$$p_1 \frac{\partial z_1}{\partial p_k} + \dots + p_{k-1} \frac{\partial z_{k-1}}{\partial p_k} + p_{k+1} \frac{\partial z_{k+1}}{\partial p_k} + \dots + p_n \frac{\partial z_n}{\partial p_k} = -p_k \frac{\partial z_k}{\partial p_k} \geq 0 \quad , \text{ since } p_k > 0 \quad .$$

$$\sum_{i \neq k} p_i \frac{\partial z_i}{\partial p_k} = -p_k \frac{\partial z_k}{\partial p_k} \geq 0 \quad , \text{ by (37)}. \quad (38)$$

Relation (38) indicates that  $\frac{\partial z_i}{\partial p_k}$  for  $k \neq i$ , in some sense are more positive than negative, and so there appear to be more substitutes than complements. If the prices of commodities scaled by a factor  $\lambda$ , i.e.,  $\mathbf{p} \rightarrow \lambda \mathbf{p}$ , the total cost is also scaled by the same factor  $\lambda$ , hence the vector functions  $\mathbf{z}(\mathbf{p})$  must remain unchanged;

$$\mathbf{z}(\lambda \mathbf{p}) = \mathbf{z}(\mathbf{p}) \quad \lambda > 0 \quad , \text{ for } \quad . \quad (39)$$

That is,  $\mathbf{z}(\mathbf{p})$  is homogeneous of degree 0 in  $\mathbf{p}$ . By Euler's Theorem that in this case amounts to setting the derivative of the left hand side of (39) with respect to  $\lambda$  equal to zero,

$$\frac{d\mathbf{z}}{d\lambda} = \sum_k \frac{\partial \mathbf{z}}{\partial (\lambda p_k)} \frac{d(\lambda p_k)}{d\lambda} = 0$$

$$\sum_k p_k \frac{\partial z_i}{\partial p_k} = 0$$

i.e.,  $\sum_k p_k \frac{\partial z_i}{\partial p_k} = 0, i = 1, 2, \dots, n.$  (40)

Obviously, (40) is same as (37).

## 6.1 Explicit Example

Now we want to discuss an explicit example to verify the Substitution and Reciprocity Theorems and the property of homogeneity of degree zero. For simplicity we consider  $n = 3$ . For the utility function  $u(\mathbf{x})$  we want to minimize the cost,

$$C = p_1 x_1 + p_2 x_2 + p_3 x_3$$

(41)

subject to the budget constraint,

$$x_1 x_2 x_3 = c_0, \text{ i.e., } x_3 = \frac{c_0}{x_1 x_2}.$$

(42)

From (41) and (42) we can write;

$$C = p_1 x_1 + p_2 x_2 + \frac{p_3 c_0}{x_1 x_2} = \hat{c}(x_1, x_2)$$

Necessary conditions for optimization are;

$$\frac{\partial \hat{c}}{\partial x_1} = p_1 - \frac{p_3 c_0}{x_1^2 x_2} = 0$$

(43a)

$$\frac{\partial \hat{c}}{\partial x_2} = p_2 - \frac{p_3 c_0}{x_1 x_2^2} = 0$$

(43b)

From (43a) we get;  $\frac{p_3 c_0}{x_1 x_2} = p_1 x_1$ , and (43b) we get;  $\frac{p_3 c_0}{x_1 x_2} = p_2 x_2$ . Therefore;

$$p_1 x_1 = p_2 x_2 = p_3 x_3$$

(44)

From (44) we get;

$$x_2 = \frac{p_1}{p_2} x_1, \text{ and} \quad (45a)$$

$$x_3 = \frac{p_1}{p_3} x_1. \quad (45b)$$

Using (45a,b) in (42) we get;

$$x_1 = \frac{c_0}{x_2 x_3} = \frac{c_0}{\frac{p_1}{p_2} x_1 \cdot \frac{p_1}{p_3} x_1} \Rightarrow x_1^3 = \frac{c_0 p_2 p_3}{p_1^2}$$

$$x_1 = z_1 = \left( \frac{c_0 p_2 p_3}{p_1^2} \right)^{\frac{1}{3}}, \text{ and similarly we get,}$$

$$x_2 = z_2 = \left( \frac{c_0 p_1 p_3}{p_2^2} \right)^{\frac{1}{3}}, \text{ and} \quad (46)$$

$$x_3 = z_3 = \left( \frac{c_0 p_1 p_2}{p_3^2} \right)^{\frac{1}{3}}.$$

Applying differentiation in (46) we get;

$$\frac{\partial z_1}{\partial p_1} = (c_0 p_2 p_3)^{\frac{1}{3}} \left( -\frac{2}{3} \right) p_1^{-\frac{5}{3}} = -\frac{2}{3} (c_0 p_2 p_3)^{\frac{1}{3}} p_1^{-\frac{5}{3}},$$

$$\frac{\partial z_2}{\partial p_2} = -\frac{2}{3} (c_0 p_1 p_3)^{\frac{1}{3}} p_2^{-\frac{5}{3}}, \text{ and} \quad (47)$$

$$\frac{\partial z_3}{\partial p_3} = -\frac{2}{3} (c_0 p_1 p_2)^{\frac{1}{3}} p_3^{-\frac{5}{3}}.$$

Relations (47) verify substitution Theorem (35). Now we calculate other derivatives as;

$$\frac{\partial z_1}{\partial p_2} = \frac{1}{3} \left( \frac{c_0 p_3}{p_1^2} \right)^{\frac{1}{3}} p_2^{-\frac{2}{3}} = \frac{1}{3} \left( \frac{c_0 p_3}{p_1^2 p_2^2} \right)^{\frac{1}{3}} = \frac{\partial z_2}{\partial p_1},$$

$$\frac{\partial z_1}{\partial p_3} = \frac{1}{3} \left( \frac{c_0 p_2}{p_1^2} \right)^{\frac{1}{3}} p_3^{-\frac{2}{3}} = \frac{1}{3} \left( \frac{c_0 p_2}{p_1^2 p_3^2} \right)^{\frac{1}{3}} = \frac{\partial z_3}{\partial p_1}, \text{ and}$$

$$\frac{\partial z_2}{\partial p_3} = \frac{1}{3} \left( \frac{c_0 p_1}{p_2^2} \right)^{\frac{1}{3}} p_3^{-\frac{2}{3}} = \frac{1}{3} \left( \frac{c_0 p_1}{p_2^2 p_3^2} \right)^{\frac{1}{3}} = \frac{\partial z_3}{\partial p_2}.$$

The relations (48) support the Reciprocity Theorem (32).

## 7. Substitution and Reciprocity Theorems with Lagrange Multipliers $\lambda$

Let us consider a single Lagrange multiplier  $\lambda$ , but it is the utility function that provides the constraint. By (41) and (42) we get;

$$K(x_1, x_2, x_3; \lambda) = p_1 x_1 + p_2 x_2 + p_3 x_3 - \lambda(x_1 x_2 x_3 - c_0)$$

For minimizations we get;

$$\frac{\partial K}{\partial x_1} = p_1 - \lambda x_2 x_3 = 0$$

$$\frac{\partial K}{\partial x_2} = p_2 - \lambda x_1 x_3 = 0$$

$$\frac{\partial K}{\partial x_3} = p_3 - \lambda x_1 x_2 = 0$$

$$\frac{\partial K}{\partial \lambda} = x_1 x_2 x_3 - c_0 = 0$$

Equation (50d) is just constraint (42). From (50a,b,c) we get the relations;

$$\lambda = \frac{p_1}{x_2 x_3} = \frac{p_2}{x_1 x_3} = \frac{p_3}{x_1 x_2}$$

$$\Rightarrow \frac{p_1}{x_2 x_3} = \frac{p_2}{x_1 x_3} \Rightarrow \frac{p_1}{x_2} = \frac{p_2}{x_1} \Rightarrow p_1 x_1 = p_2 x_2 = p_3 x_3$$

From (50a,b,c) we get;

$$\lambda x_2 x_3 = p_1 \quad , \quad (53a)$$

$$\lambda x_1 x_3 = p_2 \quad , \text{ and} \quad (53b)$$

$$\lambda x_1 x_2 = p_3 \quad . \quad (53c)$$

To eliminate  $x_1$ ,  $x_2$ ,  $x_3$ ; multiplying equations of (53) we get;

$$\lambda^3 x_1^2 x_2^2 x_3^2 = p_1 p_2 p_3 \Rightarrow \lambda^3 c_0^2 = p_1 p_2 p_3 \quad , \text{ by (50d)}$$

$$\Rightarrow \lambda = \frac{(p_1 p_2 p_3)^{1/3}}{c_0^{2/3}} \quad . \quad (54)$$

Now we consider the  $n$ -dimensional utility function;  $u(x_1, \dots, x_n) = u(x) = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$  where  $\alpha_1, \alpha_2, \dots, \alpha_n$  are positive real numbers. For this typical indifference hypersurface we have;

$$x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n} = c_0 \quad (55)$$

for a fixed value of the constant  $c_0$ . The minimized cost is given by;

$$C^{(n)} = p_1 x_1 + p_2 x_2 + \dots + p_n x_n \quad . \quad (56)$$

From (55) we get;

$$x_n = \left( \frac{c_0}{x_1^{\alpha_1} \dots x_{n-1}^{\alpha_{n-1}}} \right)^{1/\alpha_n} = \hat{c}_0 x_1^{-\alpha_1/\alpha_n} x_2^{-\alpha_2/\alpha_n} \dots x_{n-1}^{-\alpha_{n-1}/\alpha_n} \quad \hat{c}_0 = c_0^{1/\alpha_n} \quad , \text{ where} \quad . \quad (57)$$

Substituting  $x_n$  in (56) we obtain,

$$C^{(n)} = p_1 x_1 + p_2 x_2 + \dots + p_{n-1} x_{n-1} + p_n \left( \hat{c}_0 x_1^{-\alpha_1/\alpha_n} x_2^{-\alpha_2/\alpha_n} \dots x_{n-1}^{-\alpha_{n-1}/\alpha_n} \right) \equiv K^{(n-1)}(x_1, \dots, x_{n-1}) \quad . \quad (58)$$

First order differentiations for minimization we get from (58);

$$\begin{aligned}\frac{\partial K^{(n-1)}}{\partial x_1} &= p_1 - \frac{\alpha_1}{\alpha_n} p_n \hat{c}_0 x_1^{-\alpha_1/\alpha_n - 1} x_2^{-\alpha_2/\alpha_n} \dots x_{n-1}^{-\alpha_{n-1}/\alpha_n} = 0 \\ &\quad , \\ \frac{\partial K^{(n-1)}}{\partial x_2} &= p_2 - \frac{\alpha_2}{\alpha_n} p_n \hat{c}_0 x_1^{-\alpha_1/\alpha_n} x_2^{-\alpha_2/\alpha_n - 1} \dots x_{n-1}^{-\alpha_{n-1}/\alpha_n} = 0 \\ &\quad , \\ &\quad \dots \quad \dots \quad \dots \\ \frac{\partial K^{(n-1)}}{\partial x_{n-1}} &= p_{n-1} - \frac{\alpha_{n-1}}{\alpha_n} p_n \hat{c}_0 x_1^{-\alpha_1/\alpha_n} x_2^{-\alpha_2/\alpha_n} \dots x_{n-1}^{-\alpha_{n-1}/\alpha_n - 1} = 0 \\ &\quad .\end{aligned}\tag{59}$$

From the first equation of (59) we get;

$$\frac{p_1 x_1}{\alpha_1} = \frac{p_n}{\alpha_n} \hat{c}_0 x_1^{-\alpha_1/\alpha_n} x_2^{-\alpha_2/\alpha_n} \dots x_{n-1}^{-\alpha_{n-1}/\alpha_n} .$$

From the second equation of (59) we get;

$$\frac{p_2 x_2}{\alpha_2} = \frac{p_n}{\alpha_n} \hat{c}_0 x_1^{-\alpha_1/\alpha_n} x_2^{-\alpha_2/\alpha_n} \dots x_{n-1}^{-\alpha_{n-1}/\alpha_n} .$$

From the last equation of (59) we get;

$$\frac{p_{n-1} x_{n-1}}{\alpha_{n-1}} = \frac{p_n}{\alpha_n} \hat{c}_0 x_1^{-\alpha_1/\alpha_n} x_2^{-\alpha_2/\alpha_n} \dots x_{n-1}^{-\alpha_{n-1}/\alpha_n} .$$

Combining the above results we obtain;

$$\frac{p_1 x_1}{\alpha_1} = \frac{p_2 x_2}{\alpha_2} = \dots = \frac{p_{n-1} x_{n-1}}{\alpha_{n-1}} = \frac{p_n}{\alpha_n} \hat{c}_0 x_1^{-\alpha_1/\alpha_n} x_2^{-\alpha_2/\alpha_n} \dots x_{n-1}^{-\alpha_{n-1}/\alpha_n} = \frac{p_n x_n}{\alpha_n} = h \tag{say}, \text{ by (57).} \tag{60}$$

$$x_1 = \frac{\alpha_1}{p_1} h \quad x_2 = \frac{\alpha_2}{p_2} h \quad x_n = \frac{\alpha_n}{p_n} h$$

From (60) we get; , , ..., .

$$x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n} = \left( \frac{\alpha_1}{p_1} \right)^{\alpha_1} \left( \frac{\alpha_2}{p_2} \right)^{\alpha_2} \dots \left( \frac{\alpha_n}{p_n} \right)^{\alpha_n} h^{\alpha_1 + \alpha_2 + \dots + \alpha_n} = c_0$$

From (55) we get;



$$\left(\frac{\alpha_1}{p_1}\right)^{\alpha_1} \left(\frac{\alpha_2}{p_2}\right)^{\alpha_2} \dots \left(\frac{\alpha_n}{p_n}\right)^{\alpha_n} h^A = c_0 \Rightarrow h^A = c_0 \left(\frac{\alpha_1}{p_1}\right)^{-\alpha_1} \left(\frac{\alpha_2}{p_2}\right)^{-\alpha_2} \dots \left(\frac{\alpha_n}{p_n}\right)^{-\alpha_n}$$

$$h = c_0^{1/A} \left(\frac{\alpha_1}{p_1}\right)^{-\alpha_1/A} \left(\frac{\alpha_2}{p_2}\right)^{-\alpha_2/A} \dots \left(\frac{\alpha_n}{p_n}\right)^{-\alpha_n/A} = c_0^{1/A} \Lambda$$

$$\Lambda = \left(\frac{p_1}{\alpha_1}\right)^{\alpha'_1} \left(\frac{p_2}{\alpha_2}\right)^{\alpha'_2} \dots \left(\frac{p_n}{\alpha_n}\right)^{\alpha'_n} \quad \alpha'_i = \frac{\alpha_i}{A} \quad A = \alpha_1 + \alpha_2 + \dots + \alpha_n$$

with

$$\frac{p_1 x_1}{\alpha_1} = \frac{p_2 x_2}{\alpha_2} = \dots = \frac{p_n x_n}{\alpha_n} = c_0^{1/A} \Lambda$$

Hence, from (60) we get; (61)

$$x_1 = z_1(\mathbf{p}) = c_0^{1/A} \frac{\alpha_1}{p_1} \Lambda$$

From (61) we get;

$$x_2 = z_2(\mathbf{p}) = c_0^{1/A} \frac{\alpha_2}{p_2} \Lambda$$

...

$$x_n = z_n(\mathbf{p}) = c_0^{1/A} \frac{\alpha_n}{p_n} \Lambda$$

$$\frac{\partial z_1}{\partial p_1} = -c_0^{1/A} \left( \frac{\alpha_1}{p_1^2} \Lambda - \frac{\alpha_1}{p_1} \frac{\partial \Lambda}{\partial p_1} \right)$$

Differentiating (62) we get; (63)

Now differentiating  $\Lambda$  with respect to  $p_1$  we get;

$$\frac{\partial \Lambda}{\partial p_1} = \frac{\alpha'_1}{p_1} \left(\frac{p_1}{\alpha_1}\right)^{\alpha'_1} \left(\frac{p_2}{\alpha_2}\right)^{\alpha'_2} \dots \left(\frac{p_n}{\alpha_n}\right)^{\alpha'_n} = \frac{\alpha'_1}{p_1} \Lambda$$

Similarly, we can write;

$$\frac{\partial \Lambda}{\partial p_2} = \frac{\alpha'_2}{p_2} \Lambda$$

(64)

$$\frac{\partial \Lambda}{\partial p_n} = \frac{\alpha'_n}{p_n} \Lambda$$

Therefore,

$$\frac{\partial z_1}{\partial p_1} = -c_0^{1/A} \left( \frac{\alpha_1}{p_1^2} - \frac{\alpha_1 \alpha'_1}{p_1^2} \right) \Lambda = -\frac{\alpha_1 c_0^{1/A} \Lambda}{p_1^2} \frac{\alpha_2 + \alpha_3 + \dots + \alpha_n}{A}$$

Similarly we get;

$$\frac{\partial z_2}{\partial p_2} = -\frac{\alpha_2 c_0^{1/A} \Lambda}{p_2^2} \frac{\alpha_1 + \alpha_3 + \dots + \alpha_n}{A}$$

$$\dots \dots \dots (65)$$

$$\frac{\partial z_n}{\partial p_n} = -\frac{\alpha_n c_0^{1/A} \Lambda}{p_n^2} \frac{\alpha_1 + \alpha_2 + \dots + \alpha_{n-1}}{A}$$

Above relations satisfy the Substitution Theorem (35). For Reciprocity Theorem (32), we can write from (62) as;

$$\frac{\partial z_1}{\partial p_2} = c_0^{1/A} \frac{\alpha_1}{p_1} \frac{\partial \Lambda}{\partial p_2} = c_0^{1/A} \frac{\alpha_1 \alpha'_2}{p_1 p_2} \Lambda = c_0^{1/A} \frac{\alpha_1 \alpha_2}{A p_1 p_2} \Lambda$$

Similarly, we get the expression;

$$\frac{\partial z_2}{\partial p_1} = c_0^{1/A} \frac{\alpha_2}{p_2} \frac{\partial \Lambda}{\partial p_1} = c_0^{1/A} \frac{\alpha'_1 \alpha_2}{p_1 p_2} \Lambda = c_0^{1/A} \frac{\alpha_1 \alpha_2}{A p_1 p_2} \Lambda$$

$$\frac{\partial z_1}{\partial p_2} = \frac{\partial z_2}{\partial p_1} \qquad \frac{\partial z_i}{\partial p_j} = \frac{\partial z_j}{\partial p_i}$$

$$\text{Hence,} \qquad \text{So that we can write,} \qquad , \text{ for } i, j = 1, 2, \dots, n. \qquad (66)$$

Equation (66) satisfies Reciprocity Theorem (32). Again we have;

$$\Lambda(\lambda \mathbf{p}) = \left( \frac{\lambda p_1}{\alpha_1} \right)^{\alpha'_1} \left( \frac{\lambda p_2}{\alpha_2} \right)^{\alpha'_2} \dots \left( \frac{\lambda p_n}{\alpha_n} \right)^{\alpha'_n} = \lambda^{\alpha'_1} \left( \frac{p_1}{\alpha_1} \right)^{\alpha'_1} \lambda^{\alpha'_2} \left( \frac{p_2}{\alpha_2} \right)^{\alpha'_2} \dots \lambda^{\alpha'_n} \left( \frac{p_n}{\alpha_n} \right)^{\alpha'_n}$$

$$= \lambda^{\alpha'_1 + \alpha'_2 + \dots + \alpha'_n} \Lambda = \lambda \Lambda \qquad \alpha'_1 + \alpha'_2 + \dots + \alpha'_n = 1 \qquad \text{as} \qquad (67)$$

$$z_i(\lambda \mathbf{p}) = z_i(\mathbf{p})$$

Hence, using the relation (67), it is clear that,  $z_i(\mathbf{p})$ . Therefore, the homogeneity of degree zero of the  $z_i(\mathbf{p})$  is verified.

## 7.1 Explicit Example

Consider the cost (56) is minimum subject to the budget constraint (55), with the Lagrange multiplier, we get;

$$\hat{L}(x_1, \dots, x_n; \lambda) = p_1 x_1 + p_2 x_2 + \dots + p_n x_n - \lambda (x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n} - c_0) \quad (68)$$

Differentiating (68) for minimization we get;

$$\frac{\partial \hat{L}}{\partial x_1} = p_1 - \lambda (\alpha_1 x_1^{\alpha_1-1} x_2^{\alpha_2} \dots x_n^{\alpha_n}) = 0$$

$$\frac{\partial \hat{L}}{\partial x_2} = p_2 - \lambda (\alpha_2 x_1^{\alpha_1} x_2^{\alpha_2-1} \dots x_n^{\alpha_n}) = 0$$

$$\dots \dots \dots \quad (69)$$

$$\frac{\partial \hat{L}}{\partial x_n} = p_n - \lambda (\alpha_n x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n-1}) = 0$$

, and

$$\frac{\partial \hat{L}}{\partial \lambda} = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n} - c_0 = 0$$

The last equation is constraint (55). From (69) we get;

$$\frac{p_1 x_1}{\alpha_1} = \lambda x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$$

$$\frac{p_2 x_2}{\alpha_2} = \lambda x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$$

$$\dots \dots \dots$$

$$\frac{p_n x_n}{\alpha_n} = \lambda x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$$

$$\lambda = c_0^{\frac{1}{A}-1} \Lambda = c_0^{\frac{1}{A}-1} \left( \frac{p_1}{\alpha_1} \right)^{\alpha'_1} \left( \frac{p_2}{\alpha_2} \right)^{\alpha'_2} \dots \left( \frac{p_n}{\alpha_n} \right)^{\alpha'_n}$$

where

$$\frac{p_1 x_1}{\alpha_1} = \frac{p_2 x_2}{\alpha_2} = \dots = \frac{p_n x_n}{\alpha_n}$$

Hence,

(70)

which is similar to (61).

## 8. The Slutsky Equation

$$\mathbf{p} \cdot \mathbf{x} \leq k$$

Let us denote the bundle of commodities which maximizes the utility (1) subject to

by  $\zeta$  as;

$$\zeta = \zeta(\mathbf{p}, k) \quad (71)$$

$$u(\zeta) = \zeta_1 \zeta_2$$

For  $n = 2$ , we have the utility function;

Let us consider the indifference hyperbola

$$\zeta_1 \zeta_2 = c \quad (72)$$

and the straight line,

$$p_1 \zeta_1 + p_2 \zeta_2 = k \quad (73)$$

coincide at a point. From (73) we get;

$$\zeta_2 = \frac{k - p_1 \zeta_1}{p_2} \quad (74)$$

From (72) and (74) we get;

$$\zeta_1 \left( \frac{k - p_1 \zeta_1}{p_2} \right) = c_0 \Rightarrow p_1 \zeta_1^2 - k \zeta_1 + c_0 p_2 = 0$$

$$k^2 - 4 p_1 p_2 c_0 = 0 \Rightarrow k = 2(c_0 p_1 p_2)^{1/2}$$

For the coincide point,

Hence, coincide point is,

$$\zeta = (\zeta_1, \zeta_2) = \left( \frac{k}{2p_1}, \frac{k}{2p_2} \right) \quad n=3$$

For we can write the point of coincidence as,

$$\zeta = (\zeta_1, \zeta_2, \zeta_3) = \left( \frac{k}{3p_1}, \frac{k}{3p_2}, \frac{k}{3p_3} \right) \quad r(\mathbf{p})$$

and so on. We have used the cost of bundle that minimizes the

cost of indifference hypersurface (72), the minimizing vector being  $\mathbf{z}(\mathbf{p})$ , so that,  $r(\mathbf{p}) = \mathbf{p} \cdot \mathbf{z}(\mathbf{p})$ . We can write the vector  $\mathbf{z}(\mathbf{p})$  as;

$$\mathbf{z}(\mathbf{p}) = \zeta(\mathbf{p}, r(\mathbf{p})) \quad (75)$$

where  $\zeta$  is defined by (71), obviously  $k$  is replaced by  $r(\mathbf{p})$ , so that,  $r(\mathbf{p}) = 2(c_0 p_1 p_2)^{1/2}$  and  $r(\mathbf{p}) = 3(c_0 p_1 p_2 p_3)^{1/3}$  similarly for  $n = 3$ , we can write. Also we can write,

$$\begin{aligned} \mathbf{z}(\mathbf{p}) &= (z_1(\mathbf{p}), z_2(\mathbf{p})) = \left( \frac{k}{2p_1}, \frac{k}{2p_2} \right) = \left( \frac{2\sqrt{p_1 p_2 c_0}}{2p_1}, \frac{2\sqrt{p_1 p_2 c_0}}{2p_2} \right) \\ &= \left( \left( \frac{c_0 p_2}{p_1} \right)^{1/2}, \left( \frac{c_0 p_1}{p_2} \right)^{1/2} \right) \end{aligned} \quad (76)$$

Now we can define vector  $\mathbf{v}$  as;

$$\mathbf{v} = \frac{\partial \zeta}{\partial k} \quad (77)$$

Also we can write;

$$\frac{\partial \mathbf{z}}{\partial p_j} = \frac{\partial \zeta}{\partial p_j} + \frac{\partial \zeta}{\partial k} \frac{\partial r}{\partial p_j} = \frac{\partial \zeta}{\partial p_j} + \mathbf{v} \zeta_j \quad ; \text{ by (31).} \quad (78)$$

Now we are in a position to write down the Slutsky Equation as follows:

$$d\zeta_j = \sum_i V_{ji} dp_i + v_j (dk - \zeta_i dp_i) \quad (79a)$$

which can be written in the form;

$$d\zeta = V d\mathbf{p} + \mathbf{v} (dk - \zeta \cdot d\mathbf{p}) \quad (79b)$$

with  $V = (V_{ji})$  the matrix is given by;

$$V_{ji} = \frac{\partial z_j}{\partial p_i} = \left( \frac{\partial \zeta_j}{\partial p_i} \right)_{u=\text{constant}} \quad (80)$$

## 8.1 An Explicit Example

Let us consider  $n = 3$ , the utility function is given by;  $u(x_1, x_2, x_3) = x_1 x_2 x_3$ , with the budget constraint;  $p_1 x_1 + p_2 x_2 + p_3 x_3 = k$ , and  $\zeta = (\zeta_1, \zeta_2, \zeta_3) = \left( \frac{k}{3p_1}, \frac{k}{3p_2}, \frac{k}{3p_3} \right)$ . Thus,  $\mathbf{v} = \frac{\partial \zeta}{\partial k} = \left( \frac{1}{3p_1}, \frac{1}{3p_2}, \frac{1}{3p_3} \right)$  and  $C = p_1 x_1 + p_2 x_2 + p_3 x_3$ . For the minimization of the cost,  $x_1 x_2 x_3 = c_0$ , and on the indifference hypersurface,  $z_1(\mathbf{p}) = (c_0 p_2 p_3)^{1/3} p_1^{-2/3}$ ,  $z_2(\mathbf{p}) = (c_0 p_1 p_3)^{1/3} p_2^{-2/3}$ , and  $z_3(\mathbf{p}) = (c_0 p_1 p_2)^{1/3} p_3^{-2/3}$ , we get,  $r(\mathbf{p}) = p_1 z_1(\mathbf{p}) + p_2 z_2(\mathbf{p}) + p_3 z_3(\mathbf{p}) = 3(c_0 p_1 p_2 p_3)^{1/3}$ . So that;  $(V_{ji}) = \frac{dz_j}{dp_i}$ . Now we can write the matrix as follows:

$$V = (V_{ji}) = \begin{bmatrix} \frac{\partial z_1}{\partial p_1} & \frac{\partial z_1}{\partial p_2} & \frac{\partial z_1}{\partial p_3} \\ \frac{\partial z_2}{\partial p_1} & \frac{\partial z_2}{\partial p_2} & \frac{\partial z_2}{\partial p_3} \\ \frac{\partial z_3}{\partial p_1} & \frac{\partial z_3}{\partial p_2} & \frac{\partial z_3}{\partial p_3} \end{bmatrix} = \begin{bmatrix} -\frac{2}{3}(c_0 p_2 p_3)^{1/3} p_1^{-5/3} & \frac{1}{3}(c_0 p_3)^{1/3} (p_1 p_2)^{-2/3} & \frac{1}{3}(c_0 p_2)^{1/3} (p_1 p_3)^{-2/3} \\ \frac{1}{3}(c_0 p_3)^{1/3} (p_1 p_2)^{-2/3} & -\frac{2}{3}(c_0 p_1 p_3)^{1/3} p_2^{-5/3} & \frac{1}{3}(c_0 p_1)^{1/3} (p_2 p_3)^{-2/3} \\ \frac{1}{3}(c_0 p_2)^{1/3} (p_1 p_3)^{-2/3} & \frac{1}{3}(c_0 p_1)^{1/3} (p_2 p_3)^{-2/3} & -\frac{2}{3}(c_0 p_1 p_2)^{1/3} p_3^{-5/3} \end{bmatrix} \quad (48),$$

$$= \frac{1}{3} c_0^{1/3} (p_1 p_2 p_3)^{-2/3} \begin{bmatrix} -2p_1^{-1} p_2 p_3 & p_3 & p_2 \\ p_3 & -2p_1 p_2^{-1} p_3 & p_1 \\ p_2 & p_1 & -2p_1 p_2 p_3^{-1} \end{bmatrix} \quad (81)$$

In equation (80),  $u = \text{constant}$ , implies that  $\frac{c_0}{V = (V_{ij})}$  is constant with respect to  $p_i$ . According to the Reciprocity and Substitution Theorems the matrix  $\zeta$ , given by (81), is symmetric, with its diagonal elements negative. Now, we can write  $\zeta$  as a column vector as;

$$\zeta = \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{pmatrix} = \frac{k}{3} \begin{pmatrix} p_1^{-1} \\ p_2^{-1} \\ p_3^{-1} \end{pmatrix} \quad (82)$$

The differential form of (82) gives;

$$d\zeta = \frac{1}{3} \begin{pmatrix} p_1^{-1} \\ p_2^{-1} \\ p_3^{-1} \end{pmatrix} dk - \frac{k}{3} \begin{pmatrix} p_1^{-2} dp_1 \\ p_2^{-2} dp_2 \\ p_3^{-2} dp_3 \end{pmatrix} \quad (83)$$

Equation (83) provides the left hand side of the Slutsky Equation (79b). The right hand side of (79b) is;

$$\begin{aligned} V d\mathbf{p} + (dk - \zeta \cdot d\mathbf{p}) \mathbf{v} &= \frac{1}{3} c_0^{1/3} (p_1 p_2 p_3)^{-2/3} \begin{bmatrix} -2p_1^{-1} p_2 p_3 & p_3 & p_2 \\ p_3 & -2p_1 p_2^{-1} p_3 & p_1 \\ p_2 & p_1 & -2p_1 p_2 p_3^{-1} \end{bmatrix} \begin{bmatrix} dp_1 \\ dp_2 \\ dp_3 \end{bmatrix} \\ &+ \frac{1}{3} \left( dk - \frac{1}{3} k \left[ \frac{dp_1}{p_1} + \frac{dp_2}{p_2} + \frac{dp_3}{p_3} \right] \right) \begin{pmatrix} p_1^{-1} \\ p_2^{-1} \\ p_3^{-1} \end{pmatrix} \end{aligned} \quad (84)$$

$$\begin{aligned} &= \frac{1}{3} c_0^{1/3} (p_1 p_2 p_3)^{-2/3} \begin{bmatrix} -2p_1^{-1} p_2 p_3 dp_1 + p_3 dp_2 + p_2 dp_3 \\ p_3 dp_1 - 2p_1 p_2^{-1} p_3 dp_2 + p_1 dp_3 \\ p_2 dp_1 + p_1 dp_2 - 2p_1 p_2 p_3^{-1} dp_3 \end{bmatrix} + \frac{1}{3} \begin{pmatrix} p_1^{-1} \\ p_2^{-1} \\ p_3^{-1} \end{pmatrix} dk - \frac{1}{9} k \left[ \frac{dp_1}{p_1} + \frac{dp_2}{p_2} + \frac{dp_3}{p_3} \right] \begin{pmatrix} p_1^{-1} \\ p_2^{-1} \\ p_3^{-1} \end{pmatrix} \end{aligned} \quad (85)$$

The coefficient of  $dp_1$  on the right hand side of (85) is;

$$\begin{bmatrix} -\frac{2}{3}(c_0 p_2 p_3)^{1/3} p_1^{-5/3} - \frac{1}{9} \frac{k}{p_1^2} \\ \frac{1}{3}(c_0 p_3)^{1/3} (p_1 p_2)^{-2/3} - \frac{1}{9} \frac{k}{p_1 p_2} \\ \frac{1}{3}(c_0 p_2)^{1/3} (p_1 p_3)^{-2/3} - \frac{1}{9} \frac{k}{p_1 p_3} \end{bmatrix} \quad (86)$$

Since the point  $\zeta$  lies on the indifference hypersurface,  $x_1 x_2 x_3 = c_0$ , we have,

$$\frac{1}{27} \frac{k^3}{p_1 p_2 p_3} = c_0 \quad (c_0 p_1 p_2 p_3)^{1/3} = \frac{1}{3} k, \quad \text{i.e.,} \quad (87)$$

Using (87), we can write the first row of (86) as;

$$-\frac{2}{3}(c_0 p_2 p_3)^{1/3} p_1^{-5/3} - \frac{1}{9} \frac{k}{p_1^2} = -\frac{2}{3 p_1^2} (c_0 p_1 p_2 p_3)^{1/3} - \frac{1}{9} \frac{k}{p_1^2} = -\frac{2}{3 p_1^2} \cdot \frac{1}{3} k - \frac{1}{9} \frac{k}{p_1^2} = -\frac{1}{3} \frac{k}{p_1^2}$$

The second row of (86) is;

$$\frac{1}{3}(c_0 p_3)^{1/3} (p_1 p_2)^{-2/3} - \frac{1}{9} \frac{k}{p_1 p_2} = \frac{1}{3 p_1 p_2} (c_0 p_1 p_2 p_3)^{1/3} - \frac{1}{9} \frac{k}{p_1 p_2} = \frac{1}{3 p_1 p_2} \cdot \frac{1}{3} k - \frac{1}{9} \frac{k}{p_1 p_2} = 0$$

The third row of (86) is;

$$\frac{1}{3}(c_0 p_2)^{1/3} (p_1 p_3)^{-2/3} - \frac{1}{9} \frac{k}{p_1 p_3} = \frac{1}{3 p_1 p_3} (c_0 p_1 p_2 p_3)^{1/3} - \frac{1}{9} \frac{k}{p_1 p_3} = \frac{1}{3 p_1 p_3} \cdot \frac{1}{3} k - \frac{1}{9} \frac{k}{p_1 p_3} = 0$$

$$\begin{pmatrix} -\frac{1}{3} \frac{k}{p_1^2} \\ 0 \\ 0 \end{pmatrix}$$

Hence (86) reduces to,  $dp_1$ . This is the same as the term on the right hand side of (83),

$$\text{i.e., in the expression for } d\zeta \quad dp_2 \quad \begin{pmatrix} 0 \\ -\frac{1}{3} \frac{k}{p_2^2} \\ 0 \end{pmatrix} \quad dp_3 \quad \text{the coefficient of}$$



is,  $\begin{pmatrix} 0 \\ 0 \\ -\frac{1}{3} \frac{k}{p_3^2} \end{pmatrix}$ , i.e., the coefficient of  $dp_j$  is,  $-\frac{k}{3} \begin{pmatrix} p_1^{-2} \\ p_2^{-2} \\ p_3^{-2} \end{pmatrix}$ , and the coefficient of  $dk$  is,  $\frac{1}{3} \begin{pmatrix} p_1^{-1} \\ p_2^{-1} \\ p_3^{-1} \end{pmatrix}$ . Finally we

$$d\zeta = \frac{\partial \zeta}{\partial p_j} dp_j + \frac{\partial \zeta}{\partial k} dk = \left( \frac{\partial \mathbf{z}}{\partial p_j} - \mathbf{v} \zeta_j \right) dp_j + \mathbf{v} dk$$

can say, by (77) and (78).

Now we want to interpret Slutsky Equation in some detail, for this we consider the utility function

$$u(x) = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n} \quad \alpha_1, \alpha_2, \dots, \alpha_n$$

as before; with  $\alpha_1, \alpha_2, \dots, \alpha_n$  are positive constants. Now,

$$\zeta(\mathbf{p}, k) = (\zeta_1, \zeta_2, \dots, \zeta_n) = \left( \frac{\alpha_1 k}{p_1 A}, \frac{\alpha_2 k}{p_2 A}, \dots, \frac{\alpha_n k}{p_n A} \right)$$

$$\mathbf{z}(\mathbf{p}, c_0) = (z_1, z_2, \dots, z_n) = \left( c_0^{1/A} \frac{\alpha_1}{p_1} \Lambda, c_0^{1/A} \frac{\alpha_2}{p_2} \Lambda, \dots, c_0^{1/A} \frac{\alpha_n}{p_n} \Lambda \right)$$

Also, by (62) with

$$\alpha_1 + \alpha_2 + \dots + \alpha_n = A \quad \Lambda = \left( \frac{p_1}{\alpha_1} \right)^{\alpha'_1} \left( \frac{p_2}{\alpha_2} \right)^{\alpha'_2} \dots \left( \frac{p_n}{\alpha_n} \right)^{\alpha'_n} \quad \alpha'_i = \frac{\alpha_i}{A} \quad \mathbf{v} = \frac{\partial \zeta}{\partial k} = \frac{1}{A} \left( \frac{\alpha_1}{p_1}, \frac{\alpha_2}{p_2}, \dots, \frac{\alpha_n}{p_n} \right)$$

To verify Slutsky Equation, we first try to verify,

$$\mathbf{z}(\mathbf{p}, c_0) = \zeta(\mathbf{p}, r(\mathbf{p}, c_0)) \quad r(\mathbf{p}, c_0) = \mathbf{p} \cdot \mathbf{z}(\mathbf{p}, c_0)$$

, where

$$r(\mathbf{p}, c_0) = p_1 z_1 + p_2 z_2 + \dots + p_n z_n = c_0^{1/A} (\alpha_1 + \alpha_2 + \dots + \alpha_n) \Lambda = c_0^{1/A} A \Lambda$$

Now,

$$\zeta(\mathbf{p}, r(\mathbf{p}, c_0)) = \left( \frac{\alpha_1}{p_1 A} \cdot c_0^{1/A} A \Lambda, \frac{\alpha_2}{p_2 A} \cdot c_0^{1/A} A \Lambda, \dots, \frac{\alpha_n}{p_n A} \cdot c_0^{1/A} A \Lambda \right)$$

Hence,

$$= c_0^{1/A} \left( \frac{\alpha_1}{p_1}, \frac{\alpha_2}{p_2}, \dots, \frac{\alpha_n}{p_n} \right) \Lambda = \mathbf{z}(\mathbf{p}, c_0) \quad (88)$$

To verify Slutsky Equation, we now evaluate matrix  $V$ ,

$$\frac{\partial \Lambda}{\partial p_i} = \frac{\alpha'_i}{p_i} \Lambda = \frac{\alpha_i}{p_i A} \Lambda \quad (89)$$

The non-diagonal elements of  $V$  are given by;

$$\left( \frac{\partial z_j}{\partial p_i} \right)_{i \neq j} = c_0^{1/A} \frac{\alpha_j}{p_j} \frac{\partial \Lambda}{\partial p_i} = c_0^{1/A} \frac{\alpha_i \alpha_j}{p_i p_j A} \Lambda \quad (90)$$

which is symmetric, as expected. The diagonal elements of  $V$  are given by;

$$\left( \frac{\partial z_j}{\partial p_j} \right)_{\text{No summation}} = -c_0^{1/A} \frac{\alpha_j}{p_j^2} \Lambda + c_0^{1/A} \frac{\alpha_j}{p_j} \frac{\partial \Lambda}{\partial p_j} = c_0^{1/A} \frac{\alpha_j}{p_j^2} (-1 + \alpha'_j) \Lambda \quad (91)$$

$$(-1 + \alpha'_j) = -\frac{A - \alpha_j}{A} < 0$$

which is negative, as

. Now we can write matrix  $V$  as;

$$(V) = \begin{bmatrix} c_0^{1/A} \frac{\alpha_1}{p_1^2} (-1 + \alpha'_1) \Lambda & c_0^{1/A} \frac{\alpha_1 \alpha_2}{p_1 p_2 A} \Lambda & \dots & c_0^{1/A} \frac{\alpha_1 \alpha_n}{p_1 p_n A} \Lambda \\ c_0^{1/A} \frac{\alpha_1 \alpha_2}{p_1 p_2 A} \Lambda & c_0^{1/A} \frac{\alpha_2}{p_2^2} (-1 + \alpha'_2) \Lambda & \dots & c_0^{1/A} \frac{\alpha_2 \alpha_n}{p_2 p_n A} \Lambda \\ \dots & \dots & \dots & \dots \\ c_0^{1/A} \frac{\alpha_1 \alpha_n}{p_1 p_n A} \Lambda & c_0^{1/A} \frac{\alpha_2 \alpha_n}{p_2 p_n A} \Lambda & \dots & c_0^{1/A} \frac{\alpha_n}{p_n^2} (-1 + \alpha'_n) \Lambda \end{bmatrix}$$

$$= \frac{c_0^{1/A} \Lambda}{A} \begin{bmatrix} -\frac{\alpha_1}{p_1^2} (\alpha_1 - A) & \frac{\alpha_1 \alpha_2}{p_1 p_2} & \dots & \frac{\alpha_1 \alpha_n}{p_1 p_n} \\ \frac{\alpha_1 \alpha_2}{p_1 p_2} & -\frac{\alpha_2}{p_2^2} (\alpha_2 - A) & \dots & \frac{\alpha_2 \alpha_n}{p_2 p_n} \\ \dots & \dots & \dots & \dots \\ \frac{\alpha_1 \alpha_n}{p_1 p_n} & \frac{\alpha_2 \alpha_n}{p_2 p_n} & \dots & -\frac{\alpha_n}{p_n^2} (\alpha_n - A) \end{bmatrix} \quad (92)$$

$$\zeta = \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \dots \\ \zeta_n \end{pmatrix} = \frac{k}{A} \begin{pmatrix} \alpha_1 p_1^{-1} \\ \alpha_2 p_2^{-1} \\ \dots \\ \alpha_n p_n^{-1} \end{pmatrix} \quad d\zeta = \frac{1}{A} \begin{pmatrix} \alpha_1 p_1^{-1} \\ \alpha_2 p_2^{-1} \\ \dots \\ \alpha_n p_n^{-1} \end{pmatrix} dk - \frac{k}{A} \begin{pmatrix} \alpha_1 p_1^{-2} dp_1 \\ \alpha_2 p_2^{-2} dp_2 \\ \dots \\ \alpha_n p_n^{-2} dp_n \end{pmatrix}$$

Now,

and

. (93)

$$V d\mathbf{p} + (dk - \zeta \cdot d\mathbf{p}) \mathbf{v} = \frac{c_0^{1/A} \Lambda}{A} \begin{bmatrix} -\frac{\alpha_1}{p_1^2} (\alpha_1 - A) & \frac{\alpha_1 \alpha_2}{p_1 p_2} & \dots & \frac{\alpha_1 \alpha_n}{p_1 p_n} \\ \frac{\alpha_1 \alpha_2}{p_1 p_2} & -\frac{\alpha_2}{p_2^2} (\alpha_2 - A) & \dots & \frac{\alpha_2 \alpha_n}{p_2 p_n} \\ \dots & \dots & \dots & \dots \\ \frac{\alpha_1 \alpha_n}{p_1 p_n} & \frac{\alpha_2 \alpha_n}{p_2 p_n} & \dots & -\frac{\alpha_n}{p_n^2} (\alpha_n - A) \end{bmatrix} \begin{bmatrix} dp_1 \\ dp_2 \\ \dots \\ dp_n \end{bmatrix}$$

$$\begin{aligned}
 & + \frac{1}{A} \begin{pmatrix} \alpha_1 p_1^{-1} \\ \alpha_2 p_2^{-1} \\ \dots \\ \alpha_n p_n^{-1} \end{pmatrix} dk - \frac{k}{A^2} \left[ \frac{\alpha_1^2}{p_1^2} dp_1 + \frac{\alpha_2^2}{p_2^2} dp_2 + \dots + \frac{\alpha_n^2}{p_n^2} dp_n \right] \\
 & = \frac{c_0^{1/A} \Lambda}{A} \begin{bmatrix} -\frac{\alpha_1}{p_1^2} (\alpha_1 - A) dp_1 + \frac{\alpha_1 \alpha_2}{p_1 p_2} dp_2 + \dots + \frac{\alpha_1 \alpha_n}{p_1 p_n} dp_n \\ \frac{\alpha_1 \alpha_2}{p_1 p_2} dp_1 - \frac{\alpha_2}{p_2^2} (\alpha_2 - A) dp_2 + \dots + \frac{\alpha_2 \alpha_n}{p_2 p_n} dp_n \\ \dots & \dots & \dots \\ \frac{\alpha_1 \alpha_n}{p_1 p_n} dp_1 + \frac{\alpha_2 \alpha_n}{p_2 p_n} dp_2 + \dots - \frac{\alpha_n}{p_n^2} (\alpha_n - A) dp_n \end{bmatrix} \\
 & + \frac{1}{A} \begin{pmatrix} \alpha_1 p_1^{-1} \\ \alpha_2 p_2^{-1} \\ \dots \\ \alpha_n p_n^{-1} \end{pmatrix} dk - \frac{k}{A^2} \left[ \frac{\alpha_1^2}{p_1^2} dp_1 + \frac{\alpha_2^2}{p_2^2} dp_2 + \dots + \frac{\alpha_n^2}{p_n^2} dp_n \right]
 \end{aligned}$$

(94)

Now equating the coefficient of  $dp_1$  from the right hand side (RHS) of (94) we get,

$$-\frac{k}{A} \begin{pmatrix} \alpha_1 p_1^{-2} \\ 0 \\ \dots \\ 0 \end{pmatrix}$$

Similarly the coefficients of  $dp_2, dp_n$  and  $dk$  from the right hand side (RHS) of (94) are,

$$-\frac{k}{A} \begin{pmatrix} 0 \\ \alpha_2 p_2^{-2} \\ \dots \\ 0 \end{pmatrix}, \quad -\frac{k}{A} \begin{pmatrix} 0 \\ 0 \\ \dots \\ \alpha_n p_n^{-2} \end{pmatrix}, \quad \text{and} \quad \frac{1}{A} \begin{pmatrix} \alpha_1 p_1^{-1} \\ \alpha_2 p_2^{-1} \\ \dots \\ \alpha_n p_n^{-1} \end{pmatrix}$$

respectively. Hence, the RHS of (94) becomes,

$$\frac{1}{A} \begin{pmatrix} \alpha_1 p_1^{-1} \\ \alpha_2 p_2^{-1} \\ \dots \\ \alpha_n p_n^{-1} \end{pmatrix} dk - \frac{k}{A} \begin{pmatrix} \alpha_1 p_1^{-2} dp_1 \\ \alpha_2 p_2^{-2} dp_2 \\ \dots \\ \alpha_n p_n^{-2} dp_n \end{pmatrix}$$

Therefore, the Slutsky Equation is verified.

## 9. Conclusions

In this study we have applied mathematical techniques to verify Reciprocity and Substitution Theorems, and also Slutsky Equation. These three elements contribute a vital role in mathematical

economics. We have introduced some explicit examples and interpret them with some detail mathematical calculations. Here we have wanted to show the necessities of mathematics in solving the problems of economics and to flourish the three elements with mathematical device.

## References

- Ballantine, S. (1928). The lorentz reciprocity theorem for electric waves. *Proceedings of the IRE*, 16(4): 513-18.
- Cassels, J. W. S. (1981). *Economics for mathematicians*. Cambridge University Press, London, UK.
- Chipman, J. S. and Lenfant, J.-S. (2002). Slutsky's 1915 article: How it came to be found and interpreted. *History of Political Economy*, 34(3): 553-97.
- Falk, A. and Fischbacher, U. (2000). A theory of reciprocity, working paper No. 6. Institute for empirical research in economics, University of Zurich.
- Geary, R. C. (1950). A note on a constant-utility index of the cost of living. *Review of Economic Studies*, 18(1): 65-66.
- Hicks, J. R. (1956). *A revision of demand theory*. Oxford: Oxford University Press.
- Islam, J. N. (1997). *Aspects of mathematical economics and social choice theory*, In J. N. Islam (Ed.). Second Chittagong Conference on Mathematical Economics and its Relevance for Development. University of Chittagong, Bangladesh.
- Islam, J. N., Mohajan, H. K. and Moolio, P. (2009a). Political economy and social welfare with voting procedure. *KASBIT Business Journal*, 2(1): 42-66.
- Islam, J. N., Mohajan, H. K. and Moolio, P. (2009b). Preference of social choice in mathematical economics. *Indus Journal of Management & Social Sciences*, 3(1): 17-38.
- Leung, A. and Sproule, R. (2007). Using the compensating and equivalent variations to define the Slutsky equation under a discrete price change. *Economics Bulletin*, 4(11): 1-9.
- Mohajan, H. K. (2017). Aspects of mathematical economics, social choice and game theory, PhD Thesis. Jamal Nazrul Islam Research Centre for Mathematical and Physical Sciences (JNIRCMPS). University of Chittagong, Chittagong, Bangladesh.
- Nicholson, W. (2005). *Microeconomic theory*. 10<sup>th</sup> edn: Mason, Ohio: Thomson Higher Education.
- Raa, T. T. (1995). The substitution theorem. *Journal of Economic Theory*, 66(2): 632-36.
- Samuelson, P. A. (1947). *Foundations of economic analysis*. Cambridge: Harvard University Press.
- Slutsky, E. E. (1915). Sulla teoria del bilancio del consumatore, giornale degli economisti, 51, p. 1-26, English Translation by O. Ragusa as "On the theory of the budget of the consumer" in A. E. A. Readings in Price Theory by J. Stigler and K. E. Boulding (Eds.), (1952) London: George Allen and Unwin: 27-56.
- Stanca, L. (2007). Measuring indirect reciprocity: Whose back do we scratch?, working paper No. 131-December 2007. Department of Economics, University of Milan Bicocca.
- Varian, H. (2003). *Intermediate microeconomics: A modern approach*. 6<sup>th</sup> edn: New York: W.W. Norton.
- Vigier, A. (2012). *Lecture notes in microeconomics*. Department of Economics, University of Oslo, Norway.

Weber, C. (2002). Linearity, Slutsky symmetry, and a conjecture of Hicks. *Economics Bulletin*, 4(12): 1-5.